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A UNIFIED APPROACH TO Fα-CONTRACTIONS IN S-METRIC SPACE THEORY

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Abstract:

In this paper, we introduce a new class of contractive mappings called generalized Fα-contractions and establish fixed point theorems for such mappings in complete S-metric spaces. By employing the concept of α-admissibility, our results extend and unify several known fixed-point theorems. Examples are provided to illustrate the applicability and significance of the main results.

Key Words: Fixed Points, S-Metric Space, Compatible Maps, Fα-Contractions, α-Admissibility

Introduction:

Fixed point theory plays a central role in nonlinear analysis and has widespread applications across various fields such as differential equations, optimization theory, game theory, and mathematical economics. Over the years, numerous generalizations and extensions of Banach's contraction principle have been developed to accommodate a broader class of spaces and mappings. Among these generalizations, metric-type spaces such as G-metric spaces, b-metric spaces, and more recently, S-metric spaces have received considerable attention. The concept of an S-metric space, introduced by Sedghi et. al., generalizes the notion of a metric space by allowing the distance function to depend on three points instead of two. This framework has proven effective in extending classical fixed-point results and analyzing the behavior of iterative processes in more flexible topological settings. Several authors have contributed to the development of fixed-point theorems in S-metric spaces, incorporating various contraction conditions and auxiliary control functions.

Another significant development in fixed point theory is the introduction of admissibility concepts, such as α-admissibility, which was first proposed by Samet et al. This notion provides a useful framework to control the behavior of mappings, particularly when dealing with non-self or discontinuous mappings. By combining α-admissibility with different contractive conditions, many fixed-point results have been established in metric and generalized metric spaces.

Motivated by these advances, the present paper aims to introduce and study a new class of mappings, termed generalized Fα-contractions, in the context of complete S-metric spaces. Our approach blends the flexibility of the S-metric structure with the control provided by αα-admissibility and the generality of F-type contractive conditions. We establish new fixed-point theorems that generalize, unify, and improve upon several existing results in the literature.

In addition, we provide illustrative examples to demonstrate the applicability of our theorems and to highlight the advantages of our generalized framework. These results not only enrich the existing theory of fixed points in S-metric spaces but also open avenues for further research in more abstract settings and practical applications where classical assumptions may not hold.

Definitions:

1. Metric Space:

Let X be a non-empty set. A metric on X is a real function $d_{\lambda} : X \times X \to R$, which satisfies the following axioms:-

(i) $d_{\lambda}(x, y) \geq 0$ for all $x, y \in X$
(ii) $d_{\lambda}(x, y) = 0$, if and only if $x = y$
(iii) $d_{\lambda}(x, y) = d_{\lambda}(y, x)$ for all $x, y \in X$,
(iv) $d_{\lambda}(x, z) \leq d_{\lambda}(x, y) + d_{\lambda}(y, z)$ for all $x, y, z \in X$.

The ordered pair (X, d_{λ}) is called a metric space and $d_{\lambda}(x, y)$ is called the distance between x and y. The elements of X are called its points.

2. Contraction Mapping: Let

Let (X, d_{λ}) be a metric space and a mapping d_{λ}: $X \to X$ is said to be contraction mapping if there exist a real number μ with $0 \preccurlyeq \mu \prec 1$ s.t. $d_{\lambda}(\varphi(x), \varphi(y)) \preccurlyeq \mu d_{\lambda}(x, y)$ for all $x, y \in X$ and $x \neq y$, Thus, in contraction on X, the distance between the images of any two points is less than the distance between the points.

3. Compatible Mappings:

Let (X, d_{λ}) be a metric space. The mappings $\mathscr{F}$ and $\hbar$ where $\mathscr{F}: X^3 \to X$ and $\hbar : X \to X$ are said to be compatible if

$$\lim_{n\to\infty} d_{\lambda}(\hbar(\mathcal{F}(x_n, y_n, z_n)), \mathscr{F}(\hbar(x_n), \hbar(y_n), \hbar(z_n))) = 0$$

$$\lim_{n\to\infty} d_{\lambda}(\hbar(\mathcal{F}(y_n, x_n, y_n)), \mathscr{F}(\hbar(y_n), \hbar(x_n), \hbar(y_n))) = 0$$

$$\lim_{n\to\infty} d_{\lambda}(\hbar(\mathcal{F}(z_n, y_n, x_n)), \mathscr{F}(\hbar(z_n), \hbar(y_n), \hbar(x_n))) = 0$$

Whenever $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ are sequences in X such that

$$\lim_{n\to\infty} \mathcal{F}(x_n, y_n, z_n) = \lim_{n\to\infty} \hbar(x_n) = x$$

$$\lim_{n\,-\!\!\to\infty} \mathcal{F}(y_n, x_n, y_n) = \lim_{n\to\infty} \hbar(y_n) = y \text{ and}$$

$$\lim_{n\to\infty} \mathcal{F}(z_n, y_n, x_n) = \lim_{n\to\infty} \hbar(z_n) = z \text{ for some } x, y, z \in X.$$

4. S-Metric Space:

Let X be a non-empty set. An S - metric on X is a function $S : X^3 \to [\,0, \infty)$ that satisfies the following conditions for all x, y, z, a ∈ X.

- $S(x, y, z) = 0$ if and only if $x = y = z$.
- $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$.

The pair (X, S) is called an S - metric space.

5. Complete s-Metric Space:

Let (X, S) be an S-metric space.

- A sequence $\{x_n\}$ c X converges to x ∈ X if $S(x_n, x_n, x) \to 0$ as $n \to \infty$. That is, for each $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$. We have, $S(x_n, x_n, x) < \epsilon$. We write $x_n \to x$ for brevity.
- A sequence $\{x_n\}$ c X converges to x is a Cauchy sequence if $S(x_n, x_n, x_m) \to 0$ as $n,m \to \infty$. That is for each $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n, m \geq n_0$ we have $S(x_n, x_n, x_m) < \epsilon$. The S-metric space (X, S) is complete if every Cauchy sequence is a convergent sequence.

6. Compatible:

Let $(\mathcal{J}, S)$ be an S-metric space. A pair {l, k} is said to be compatible if and only if $\lim_{n\to\infty} S(lku, lku, lku) = 0$, whenever $\{u_n\}$ is a sequence in $\mathcal{J}$ such that$\lim_{n\to\infty} lu = \lim_{n\to\infty} ku = r$ for some $r \in \mathcal{J}$.

Let (X, S) be an S-metric space.

- A sequence $\{x_n\}$ c X converges to x ∈ X if $S(x_n, x_n, x) \to 0$ as $n \to \infty$. That is, for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we have $S(x_n, x_n, x) < \varepsilon$. We write for $x_n \to x$.
- A sequence $\{x_n\}$ c X is a Cauchy sequence if $S(x_n, x_n, x_m) \to 0$ as $n, m \to \infty$.That is, for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n, m \geq n_0$ we have $S(x_n, x_n, x_m) < \varepsilon$.

The S-metric space (X, S) is complete if every Cauchy sequence is convergent.

7. α - Admissible:

Let (X, S) be a S- metric space and $T : X \to X$ be a given mapping. We say that T is α - admissible if $\alpha(x, y, z) \geq 1 \Rightarrow \alpha(Tx, Ty , Tz) \geq 1$, for all x, y, z ∈ X

8. Let $F: \mathbb{R}^+ \to \mathbb{R}$ be a mapping satisfying:

- (F_1) F is strictly increasing.
- (F_2) For every sequence $\{x_n\}$ c $\mathbb{R}^+$,we have,$\lim_{n\to\infty} \alpha_n = 0 \Leftrightarrow \lim_{n\to\infty} F(\alpha_n) = -\infty$.
- (F_3) There exists a number k∈ (0, 1) such that $\lim_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$.

In what follows, $\mathfrak{F}$ stands for the family of all functions F which satisfies the above three conditions.

9. F- Contraction:

Let (X, S) be an S- metric space. A mapping $T: X \to X$ is said to be an F - contraction if there is a number $\tau > 0$ and an $F \in \mathfrak{F}$ such that $S(Tx, Ty, Tz) > 0 \Rightarrow \tau + F(S(Tx, Ty, Tz)) \leq F(S(x, y, z))$, for all x, y, z ∈ X.

10. F_α- Contraction:

Let (X, S) be an S-metric space and let $\alpha : X \times X \times X \to [0, \infty)$ be a function . A mapping $T : X \to X$ is called an F_α-contraction, if there exist $F \in \mathfrak{F}$ and $\tau > 0$ such that $S(Tx, Ty, Tz) > 0 \Rightarrow \tau + F(\alpha(x, y, z)\, S(Tx, Ty, Tz)) \leq F(S(x, y, z))$, for all x, y, z ∈ X.

11. α - Admissible Mapping:

Let $T: X \to X$ and $\alpha, \eta : X \times X \times X \to [0, +\infty)$ be two functions. We say that T is an α - admissible mapping with respect to η, if for all x, y ∈ X such that $\alpha(x, x, y) \geq \eta(x, ,x, y)$, then we have $\alpha(Tx, Tx, Ty) \geq \eta(Tx, Tx, Ty)$. If we take $\eta(x, x, y) = 1$, then T is called an α - admissible mapping. If we take $\alpha(x, x, y) = 1$, then T is called - sub-admissible.

Main Results:

Theorem 1:

Let (X, S) be a complete S-metric space and let $T: X \to X$ be an f_α- contraction, satisfying the following conditions:

- T is α-admissible;
- there exists $x_0 \epsilon X$ such that $\alpha(x_0, x_0, Tx_0) \geq 1$;
- T is continuous.

Then, T has a fixed point.

Proof:

Let $x_0 \epsilon X$. Consider the sequence $\{x_n\}$ defined by $x_1 = Tx_0$, $x_2 = Tx_1 = T^2x_0, \ldots, x_n = Tx_{n-1} = T^nx_0$.

By our assumption (ii), we know that $\alpha(x_0, x_0, Tx_0) \geq 1$ and as T is α-admissible, so, $\alpha(x_1, x_1, x_2) \geq 1$ and by induction on n, we conclude that $\alpha(x_n, x_n, x_{n+1}) \geq 1$, for all n.

Now, $S(x_n, x_n, x_{n+1}) = S(Tx_{n-1}, Tx_{n-1}, Tx_n) \leq \alpha(x_{n-1}, x_{n-1}, x_n)S(Tx_{n-1}, Tx_{n-1}, Tx_n)$

$\Rightarrow F(S(x_n, x_n, x_{n+1})) \leq F(\alpha(x_{n-1}, x_{n-1}, x_n)\, S(Tx_{n-1}, Tx_{n-1}, Tx_n))$.

So, $F(S(x_n, x_n, x_{n+1})) \leq F(S(x_{n-1}, x_{n-1}, x_n)) - \tau$ (1.1)

Now, $S(x_{n-1}, x_{n-1}, x_n) = S(Tx_{n-2}, Tx_{n-2}, Tx_{n-1}) \leq \alpha(x_{n-2}, x_{n-2}, x_{n-1})\, S(Tx_{n-2}, Tx_{n-2}, Tx_{n-1})$

Thus,

$F(S(x_{n-1}, x_{n-1}, x_n)) \leq F(\alpha(x_{n-2}, x_{n-2}, x_{n-1})\, S(Tx_{n-2}, Tx_{n-2}, Tx_{n-1}))$

Then, $F(S(x_{n-1}, x_{n-1}, x_n)) \leq F(S(x_{n-2}, x_{n-2}, x_{n-1})) - \tau$ (1.2)

Putting (1.2) in (1.1) we have

$F(S(x_n, x_n, x_{n+1})) \leq F(S(x_{n-2}, x_{n-2}, x_{n-1})) - 2\tau$

Continuing in this way, we get,

$F(S(x_n, x_n, x_{n+1})) \leq F(S(x_0, x_0, x_1)) - n\tau$ (1.3)

Now, let $\partial_n = S(x_n, x_n, x_{n+1})$

So, $F(\partial_n) \leq F(\partial_0) - n\tau$ (1.4)

Now, $\lim\limits_{n\to\infty} F(\partial_n) \leq \lim\limits_{n\to\infty}(F(\partial_0) - n\tau) \Longrightarrow \lim\limits_{n\to\infty} F(\partial_n) \leq -\infty \Longrightarrow \lim\limits_{n\to\infty} (\partial_n) = 0$

So, $\lim\limits_{n\to\infty} S(x_n, x_n, x_{n+1}) = 0.$ (1.5)

Now, there exists $k \in (0, 1)$ such that $\lim\limits_{n\to\infty} (\partial_n)^k F(\partial_n) = 0$ (1.6)

Now, using (1.4) implies that $\partial_n^k F(\partial_n) \leq \partial_n^k (F(\partial_0) - n\tau)$

Adding and subtracting $\partial_n^k F(\partial_0)$ on left side of the above inequality, we get

$\partial_n^k F(\partial_n) - \partial_n^k F(\partial_0) + \partial_n^k F(\partial_0) \leq \partial_n^k (F(\partial_0) - n\tau)$

$\Longrightarrow \partial_n^k F(\partial_n) - \partial_n^k F(\partial_0) \leq -\partial_n^k n\tau$

So, $\partial_n^k F(\partial_n) \leq \partial_n^k F(\partial_0) - \partial_n^k n\tau.$ (1.7)

Now, by applying limit on both sides, we have

$\lim\limits_{n\to\infty} \partial_n^k F(\partial_n) \leq \lim\limits_{n\to\infty} \partial_n^k F(\partial_0) - \lim\limits_{n\to\infty} \partial_n^k n\tau \Longrightarrow 0 = 0 - \lim\limits_{n\to\infty} \partial_n^k n\tau \Longrightarrow \lim\limits_{n\to\infty} \partial_n^k n\tau = 0. \Longrightarrow \partial_n^k n\tau \leq 1$

For $\tau = 1$, $\partial_n^k \leq \frac{1}{n}$, $\Longrightarrow [\partial_n^k]^{\frac{1}{k}} \leq \left(\frac{1}{n}\right)^{\frac{1}{k}} \Longrightarrow \partial_n \leq \frac{1}{n^{\frac{1}{k}}}$.

Now, consider $m, n \in \mathbb{N}$ and $m, n \geq n_0$, for more n_0. Then

$S(x_m, x_m, x_n) \leq \partial_{m-1} + \partial_{m-2} + \ldots\ldots + \partial_n$

$\leq \sum_{i=n}^{\infty} \partial_i \leq \sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{k}}} \leq \infty.$

Since, the series is convergent, the sequence $\{x_n\}$ is convergent, i.e., $\lim\limits_{n\to\infty} x_n = x^*$. Since T is continuous, we have

$x^* = \lim\limits_{n\to\infty} x_{n+1} = \lim\limits_{n\to\infty} Tx_n = Tx^* \Longrightarrow Tx^* = x^*.$

So, T has a fixed point.

Example 1:

Let $X = \mathbb{R}$ and $S(x, y,z) = |x - z| + |y - z|$ be an S-metric space on X. Suppose that

$\alpha(x, y, z) = \begin{cases} e^{\max\{x,y\}-z}, & \text{if } \max\{x, y\} \geq z, \\ 0, & \text{if } \max\{x, y\} < z, \end{cases}$ and $F(x) = \frac{1}{2}\sin hx$, $T(x, y, z) = 0.1$.

We have to show that $\tau + F(\alpha(x, y, z)S(Tx, Ty, Tz)) \leq F(S(x, y, z)).$ (1.8)

Take the right side of (1.8).

Let $x = 0.3$, $y = 0.2$, $z = 0.1$. We have $S(0.3, 0.2, 0.1) = = |0.3 - 0.1| + |0.2 - 0.1| = 0.3.$

Now, $F(0.3) = \frac{1}{2}\sin h(0.3) \Longrightarrow F(0.3)\ 0.3045202934456.$

Also, $\alpha(0.3, 0.2, 0.1) = e^{\max\{0.3,0.2\}-0.1} = e^{0.2} = 1.22.$

Now, we have $T(0.3) = 0.1$, $T(0.2) = 0.1$ and $T(0.1) = 0.1$. So, $S(0.1, 0.1, 0.1) = 0.$

Then, $F((1.22)(0)) = F(0) = \frac{1}{2}\sin h(0) = 0$

Putting the values in (1.8), $0.001 + 0 \leq 0.3045202934456 \Longrightarrow 0.001 \leq 0.3045202934456.$

So, T is an F_α - contraction. Now, we will show that T is α - admissible. Note that $\alpha(Tx, Ty, Tz) = e^{\max\{0.1,0.1\}-0.1} = e^0 = 1.$

So, T is α- admissible. Now, let $x_0 = 1 \in X = \mathbb{R}$.

$\alpha(1,1,0.1) = e^{\max\{1,1\}-0.1} = e^{0.9} = 2.44 \geq 1.$

Also, T is continuous, because $T(x, y, z) = 0.1$. so, T has a fixed point.

Theorem 2:

Let (X, S) be a complete S-metric space and $T : X \rightarrow X$ be a mapping. Suppose that there exist two functions $\alpha, \eta : X \times X \times X \rightarrow [0, +\infty)$ such that T is α- admissible with respect to η. Let $r > 0$, $x_0 \in B(x_0, r)$ and $\psi \in \Psi$. Assume that $x, y \in \overline{B(x_0, r)}$, $\alpha(x, x, y) \geq \eta(x, x, y) \Longrightarrow S(Tx, Tx, Ty) \leq \psi S(x, x, y)$ (1.9)

and $2\sum_{i=0}^{j} \psi^i (S(x_0, x_0, Tx_0)) \leq r$ for all $j \in \mathbb{N}$. (1.10)

Suppose that the following assertion hold :

- $\alpha(x_0, x_0, Tx_0) \geq \eta(x_0, x_0, Tx_0)$
- for any sequence $\{x_n\}$ in $B(x_0,r)$ such that $\alpha(x_n, x_n, x_{n+1}) \geq \eta(x_n,x_n, x_{n+1})$ for all $n \in N \cup \{0\}$ and $\{x_n\} \rightarrow u \in B(x_0, r)$ as $n \rightarrow +\infty$, then $\alpha(x_n, x_n, u) \geq \eta(x_n,x_n,u)$ for all $n \in N \cup \{0\}$.

Then, there is a point x^* in $B(x_0, r)$ such that $x^* = Tx^*$.

Proof:

Let $x_0 \in X$ be such that $x_1 = Tx_0$, $x_2 = Tx_1 = T(Tx_0) = T^2x_0$.

Continuing in this way we get $x_{n+1} = Tx_n$.

By assumption, $\alpha(x_0, x_0, x_1) \geq \eta(x_0, x_0, x_1)$

And as T is α- admissible with respect to η, so we have

$\alpha(Tx_0, Tx_0, Tx_1) \geq \eta(Tx_0, Tx_0, Tx_1)$.

From which we can deduce that $\alpha(x_1, x_1, x_2) \geq \eta(x_1, x_1, x_2)$, which also implies that

$\alpha(Tx_1, Tx_1, Tx_2) \geq \eta(Tx_1, Tx_1, Tx_2)$.

Continuing in this way, we get $\alpha(x_n, x_n, x_{n+1}) \geq \eta(x_n, x_n, x_{n+1})$ for all $n \in N \cup \{0\}$.

First, we will show that $x_n \in \overline{B(x_0, r)}$, for all $n \in \mathbb{N}$. Using inequality (1.10) we have

$S(x_0, x_0, Tx_0) \leq r$.

It follows that $x_1 \in \overline{B(x_0, r)}$.

Let $x_1, \ldots, x_j \in \overline{B(x_0, r)}$ for some $j \in \mathbb{N}$. If $j = 2i + 1$, where $i = 0, 1, 2, 3, \ldots. \frac{j-1}{2}$ then using in equality (1.9), we obtain

$S(x_{2i+1}, x_{2i+1}, x_{2i+2}) = S(Tx_{2i}, Tx_{2i}, Tx_{2i+1}) \leq \psi\, (x_{2i-1}, x_{2i-1}, x_{2i}) \leq \psi^2(S(x_{2i-2}, x_{2i-2}, x_{2i-1})) \leq \psi^{(2i+1)}S(x_0, x_0, x_1)$.

Thus, we have $S(x_{2i+1}, x_{2i+1}, x_{2i+2}) \leq \psi^{(2i+1)}S(x_0, x_0, x_1)$ (1.11)

If $j = 2i + 2$, then as $x_1, x_2, \ldots, x_j \in \overline{B(x_0, r)}$, where ($i = 0, 1, 2, \ldots \frac{j-1}{2}$),

we obtain, $S(x_{2i+2}, x_{2i+2}, x_{2i+3}) \leq \psi^{2(i+1)}S(x_0, x_0, x_1)$. (1.12)

Thus, from inequalities (1.11) and (1.12) we have, $S(x_j, x_j, x_{j+1}) \leq \psi^j S(x_0, x_0, x_1)$ (1.13)

Now, $S(x_0, x_0, x_{j+1}) \leq 2S(x_0, x_0, x_1) + 2S(x_1, x_1, x_2) + \ldots + 2S(x_j, x_j, x_{j+1}) \leq 2\sum_{i=0}^{j} \psi^{i}\, (S(x_0, x_0, x_1)) \leq r$

Thus $x_{j+1} \in \overline{B(x_0, r)}$. Hence $x_n \in \overline{B(x_0, r)}$, for all $n \in \mathbb{N}$. Now, inequality (1.13) can be written as $S(x_n, x_n, x_{n+1}) \leq \psi^n S(x_0, x_0, x_1)$ (1.14) for all $n \in \mathbb{N}$.

Fix $\varepsilon > 0$ and let $n(\varepsilon) \in \mathbb{N}$ such that $\sum \psi^{n}\, (S(x_0, x_0, x_1)) < \varepsilon$.

Let $n, m \in \mathbb{N}$ with $m > n > n(\varepsilon)$, then by using the triangular inequality, we obtain

$S(x_n, x_n, x_m) \leq 2\sum_{k=n}^{m-1} S(x_k, x_k, x_{k+1})$

$\leq 2\sum_{k=n}^{m-1} \psi^{k}(S(x_0, x_0, x_1)) \leq \sum_{k \geq n(\varepsilon)} \psi^{k}\, (S(x_0, x_0, x_1)) \leq \varepsilon$

Thus, we proved that $\{x_n\}$ is Cauchy sequence in $(\overline{B(x_0, r)}, S)$. as every closed ball in a complete S-metric space is complete, so there exists $x^* \in \overline{B(x_0, r)}$ such that $x_n \to x^*$. Also $\lim_{n\to\infty} S(x_n, x_n, x^*) = 0$ (1.15)

On the other hand from (ii), we have

$\alpha\, (x^*, x^*, x_n) \geq \eta(x^*, x^*, x_n)$ for all $n \in N \cup \{0\}$

Now, using the triangular inequality, together with (1.9) and (1.16) we get,

$S(Tx^*, Tx^*, x_{2i+1}) \leq \psi\, (S(x^*, x^*, x_{2i})) \leq S(x^*, x^*, x_{2i})$.

So, we obtain that $S(Tx^*, Tx^*, x^*) = 0$, that is $Tx^* = x^*$. Hence, T have a fixed point in $\overline{B(x_0, r)}$.

If we take $\eta(x, x, y) = 1$, for all $x, y \in X$, in the above result, we obtain the following result.

Corollary 1:

Let (X, S) be a complete S-metric space and $T : X \to X$, $r > 0$ and x_0 be an arbitrary point in $\overline{B(x_0, r)}$. Suppose that there exist $\alpha : X \times X \times X \to [0, +\infty)$ such that T is α- admissible. For $\psi \in \Psi$,

Assume that $x, y \in \overline{B(x_0, r)}$, $\alpha(x, x, y) \geq 1 \Rightarrow S(Tx, Tx, Ty) \leq \psi(S(x, x, y))$ (1.17) and $2\sum_{i=0}^{j} \psi^{i}\, (S(x_0, x_0, Tx_0)) \leq r$ for all $j \in \mathbb{N}$. (1.18) Suppose that the following assertions hold:

- $\alpha\, (x_0, x_0, Tx_0) \geq 1$.
- for any sequence $\{x_n\}$ in $\overline{B(x_0, r)}$ such that $\alpha(x_n, x_n, x_{n+1}) \geq 1$ for all $n \in N \cup \{0\}$ and $\{x_n\} \to u \in \overline{B(x_0, r)}$ as $n \to +\infty$, then $\alpha(x_n, x_n, u) \geq 1$ for all $n \in N \cup \{0\}$.

Then, there is a point x^* in $\overline{B(x_0, r)}$ such that $x^* = Tx^*$.

Corollary 2:

Let (X, S) be a complete S-metric space and $T: X \to X$ be a mapping. Suppose that there exist $\eta: X \times X \times X \to [0, +\infty)$ such that T is η-sub- admissible. For $\psi \in \Psi$, and x_0 be an arbitrary point in $\overline{B(x_0, r)}$.

Assume that $x, y \in \overline{B(x_0, r)}$, $\eta\, (x, x, y) \leq 1 \Rightarrow S(Tx, Tx, Ty) \leq \psi(S(x, x, y))$ and $2\sum_{i=0}^{j} \psi^{i}\, (S(x_0, x_0, Tx_0)) \leq r$ for all $j \in \mathbb{N}$. Suppose that the following assertions hold:

- $\eta(x_0, x_0, Tx_0) \leq 1$.
- for any sequence $\{x_n\}$ in $\overline{B(x_0, r)}$ such that $\eta(x_n, x_n, x_{n+1}) \leq 1$ for all $n \in N \cup \{0\}$ and $\{x_n\} \to u \in \overline{B(x_0, r)}$ as $n \to +\infty$, then $\eta(x_n, x_n, u) \leq 1$ for all $n \in N \cup \{0\}$

Then, there is a point x^* in $\overline{B(x_0, r)}$ such that $x^* = Tx^*$.

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