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SOME APPLICATIONS OF FINE'S THEOREM

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Abstract:

We exhibit how the Fine's theorem allows determine $p(5n+4)$ and $r_2(n)$ if we know the corresponding partitions of n.

Key Words: Fine's Theorem, Partition Function, Sums of Two Squares.

1. Introduction:

We have the Fine's theorem [1-3]:

If $\psi_j(q) = \sum_{n=0}^{\infty} C_j(n)\, q^n$ then $\prod_{j=1}^{\infty} \psi_j(q^j) = \sum_{n=0}^{\infty} R(n)\, q^n$, (1)

With: $R(n) = \sum_{\lambda \vdash n} C_1(k_1) C_2(k_2) \cdots C_n(k_n)$, (2)

Where $\lambda \vdash n$ means all partitions of n, and k_r is the multiplicity of r in a given partition.

In Sec. 2 we apply (1) and (2) to Ramanujan's "most beautiful identity" [4]:

$$\sum_{n=0}^{\infty} p(5n+4)\, q^n = 5 \prod_{j=1}^{\infty} \frac{(1-q^{5j})^5}{(1-q^j)^6}, \tag{3}$$

To exhibit how the partitions of n allow to obtain the value of $p(5n+4)$. In Sec. 3 we use this Fine's theorem to study the relation:

$$\sum_{n=0}^{\infty} (-1)^n\, r_2(n) q^n = \prod_{j=1}^{\infty} \left(\frac{1-q^j}{1+q^j}\right)^2, \tag{4}$$

Where $r_2(n)$ is the number of representations of n as a sum of two squares [5-7]

2. Fine's Theorem Applied to Ramanujan's Identity (3):

First we consider the functions:

$$\psi_j(q) := (1-q^5)^5\ , \tag{5}$$

And only are different to zero the following quantities:

$$C_j(5m) = (-1)^m (5m)! \binom{5}{m}, \quad m = 0, 1, \ldots, 5, \tag{6}$$

Then from (1):

$$\prod_{j=1}^{\infty} \psi_j(q^j) = \prod_{j=1}^{\infty} (1-q^{5j})^5 = \sum_{n=0}^{\infty} R(n)\, q^n, \quad R(0) = 1, \tag{7}$$

Where $R(n)$ is given by (2) and (6). Similarly, we introduce the functions:

$$\tilde{\psi}_j(q) = \frac{1}{(1-q)^6} = \sum_{n=0}^{\infty} \tilde{C}_j(n)\, q^n \quad \therefore \quad \tilde{C}_j(r) = \binom{r+5}{5}, \quad r \geq 0, \tag{8}$$

Such that:

$$\prod_{j=1}^{\infty} \tilde{\psi}_j(q^j) = \prod_{j=1}^{\infty} \frac{1}{(1-q^j)^6} = \sum_{n=0}^{\infty} \tilde{R}(n)\, q^n, \quad \tilde{R}(0) = 1, \tag{9}$$

and $\tilde{R}(n)$ can be constructed with (2) and (8). Hence, from (3), (7) and (9) we deduce that $p(5n+4)$ is the Cauchy convolution [8] of $R(n)$ with $\tilde{R}(m)$, in fact:

$$p(5n+4) = 5 \sum_{k=0}^{n} R(k) \tilde{R}(n-k), \quad n \geq 0, \tag{10}$$

Therefore $p(4) = 5,\ p(9) = 30,\ p(14) = 135, \ldots$; for example, if we know explicitly the 7 partitions of 5, then with (10) we can obtain that $p(29) = 4565$.

3. Sums of Two Squares:

Here we use the following functions:

$$\psi_j(q) = (1-q)^2 \quad \therefore \quad C_j(0) = 1, \quad C_j(1) = -2, \quad C_j(2) = 1, \quad C_j(m) = 0, \quad m \geq 3, \tag{11}$$

Then:

$$\prod_{j=1}^{\infty} \psi_j(q^j) = \prod_{j=1}^{\infty} (1-q^j)^2 = \sum_{n=0}^{\infty} R(n)\, q^n, \quad R(0) = R(4) = -R(2), \quad R(3) = -R(1) = 2, \ldots \tag{12}$$

With the participation of (2) and (11). In analogous manner, we consider the functions:

$$\tilde{\psi}_j(q) = \frac{1}{(1+q)^2} \quad \therefore \quad \tilde{C}_j(n) = (-1)^n (n+1), \quad n \geq 0, \tag{13}$$

Thus:

$$\prod_{j=1}^{\infty} \tilde{\psi}_j(q^j) = \prod_{j=1}^{\infty} \frac{1}{(1+q^j)^2} = \sum_{n=0}^{\infty} \tilde{R}(n)\, q^n, \quad \tilde{R}(0) = \tilde{R}(2) = 1, \tilde{R}(1) = \tilde{R}(3) = -2, \tilde{R}(4) = 4, \ldots \tag{14}$$

Where (2) and (13) were applied. Hence, from (4), (12) and (14):

$$r_2(n) = (-1)^n \sum_{k=0}^{n} R(k)\, \tilde{R}(n-k), \quad n \geq 0, \tag{15}$$

Then $r_2(0) = 1,\ r_2(1) = r_2(2) = r_2(4) = 4,\ r_2(3) = 0, \ldots$

The Fine's theorem permits write many arithmetic functions in terms of integer partitions.

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