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A GENERAL METHOD FOR INVERTING ARBITRARY VANDERMONDE MATRICES

R. Sivaraman*, J. López-Bonilla & S. Vidal-Beltrán****
* Department of Mathematics, Dwaraka Doss Goverdhan Doss Vaishnav College, Chennai, Tamil Nadu, India
** ESIME-Zacatenco, Instituto Politécnico Nacional, Edif. 4, 1er. Piso, Col. Lindavista CP 07738, CDMX, México

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Abstract:

Vandermonde matrices play a fundamental role in several areas of applied mathematics, including polynomial interpolation, numerical analysis, approximation theory, and signal processing. The computation of their inverse is therefore an important problem with numerous theoretical and computational applications. In this paper, we present a simple and systematic method for constructing the inverse of an arbitrary Vandermonde matrix with distinct parameters. The proposed approach is based on the characteristic polynomials of suitably defined diagonal matrices obtained by excluding one parameter at a time. This formulation provides explicit expressions for the entries of the inverse matrix without relying on recursive procedures or computationally intensive algorithms. The method is applicable to Vandermonde matrices of arbitrary order and offers an alternative to several classical inversion techniques reported in the literature. Furthermore, the proposed formulation naturally reproduces the inverse matrices of lower-dimensional cases and establishes a direct connection with previously published results. Owing to its simplicity, generality, and algebraic transparency, the proposed method provides an effective framework for the analytical inversion of Vandermonde matrices and may be useful in applications involving interpolation, matrix computations, and related numerical methods.

Key Words: Vandermonde Matrices, Characteristic Polynomial, Diagonal Matrices.

1. Introduction:

The Vandermonde matrices have the structure [1-3]:

$$V_2 = \begin{pmatrix} 1 & x_1 \\ 1 & x_2 \end{pmatrix}, V_3 = \begin{pmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{pmatrix}, V_4 = \begin{pmatrix} 1 & x_1 & x_1^2 & x_1^3 \\ 1 & x_2 & x_2^2 & x_2^3 \\ 1 & x_3 & x_3^2 & x_3^3 \\ 1 & x_4 & x_4^2 & x_4^3 \end{pmatrix}, \quad \text{etc.} \tag{1}$$

Which are important, for example, in the Lagrange interpolation [4]. It is well known that the determinants of (1) are different to zero if the x_k are distinct among them, then under this last assumption there exists their inverses. Thus, here we consider matrices of the form:

$$V_n = \begin{pmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{pmatrix}, \qquad n = 2, 3, 4, \ldots, \tag{2}$$

And in Sec. 2 we exhibit an approach to construct the inverse of a matrix of the type (2), that is, our aim is to show a method to obtain V_n^{-1}, $n = 2, 3, \ldots$, as an alternative to the algorithms presented in [5-10].

2. Inverse of an Arbitrary Vandermonde Matrix:

In fact: $V_n^{-1} = (A_{jk})$, $A_{jk} = \frac{1}{B_k} a_{jk}$, $a_{nk} = 1$, $B_k \equiv \prod_{r=1,\ r\neq k}^{n} (x_k - x_r)$, (3)

And to determine the elements a_{jk} we introduce the following $(n-1)x(n-1)$ diagonal matrices:

$$D_j = \text{Diag}\,(x_1, x_2, \ldots, x_n), \qquad j = 1, 2, \ldots, n, \tag{4}$$

Where it was removed the value x_j ; then we obtain the characteristic polynomial [2, 3, 11-13] of each D_j:

$$\lambda^{n-1} + a_{n-1,j}\lambda^{n-2} + a_{n-2,j}\lambda^{n-2} + \cdots + a_{2j}\,\lambda + a_{1j}\,, \tag{5}$$

Where we can read the values of all a_{jk} .

Now we shall apply our expressions (3), (4) and (5) to the matrices (1):

a). $n = 2$: $B_1 = -B_2 = x_1 - x_2$, $D_1 = (x_2)$, $\quad D_2 = (x_1)$, $\lambda - x_2$, $\quad \lambda - x_1$,

And the comparison with (5) gives the values $a_{11} = -x_2$ and $a_{12} = -x_1$ with $a_{21} = a_{22} = 1$, therefore:

$$V_2^{-1} = \begin{pmatrix} -\frac{x_2}{B_1} & -\frac{x_1}{B_2} \\ \frac{1}{B_1} & \frac{1}{B_2} \end{pmatrix}. \tag{6}$$

b). $n = 3$: $B_1 = (x_1 - x_2)(x_1 - x_3)$, $B_2 = (x_2 - x_1)(x_2 - x_3)$, $B_3 = (x_3 - x_1)(x_3 - x_2)$,

$$\mathrm{D}_1 = \mathrm{Diag}\,(\mathrm{x}_2\,, \mathrm{x}_3), \mathrm{D}_2 = \mathrm{Diag}\,(\mathrm{x}_1\,, \mathrm{x}_3), \mathrm{D}_3 = \mathrm{Diag}\,(\mathrm{x}_1\,, \mathrm{x}_2),$$

$$\lambda^2 - (\mathrm{x}_2 + \mathrm{x}_3)\,\lambda + \mathrm{x}_2\,\mathrm{x}_3\,, \lambda^2 - (\mathrm{x}_1 + \mathrm{x}_3)\,\lambda + \mathrm{x}_1\,\mathrm{x}_3\,, \lambda^2 - (\mathrm{x}_1 + \mathrm{x}_2)\,\lambda + \mathrm{x}_1\,\mathrm{x}_2\,,$$

Hence $a_{21} = -(x_2 + x_3)$, $a_{11} = x_2\,x_3$, $a_{22} = -(x_1 + x_3)$, $a_{12} = x_1\,x_3$, $a_{23} = -(x_1 + x_2)$, $a_{13} = x_1\,x_2$ with $a_{31} = a_{32} = a_{33} = 1$, then:

$$V_3^{-1} = \begin{pmatrix} \frac{x_2x_3}{B_1} & \frac{x_1x_3}{B_2} & \frac{x_1x_2}{B_3} \\ -\frac{x_2+x_3}{B_1} & -\frac{x_1+x_3}{B_2} & -\frac{x_1+x_2}{B_3} \\ \frac{1}{B_1} & \frac{1}{B_2} & \frac{1}{B_3} \end{pmatrix}. \tag{7}$$

c). Similarly, for $n = 4$ our approach implies the expression:

$$V_4^{-1} = \begin{pmatrix} -\frac{x_2x_3x_4}{B_1} & -\frac{x_1x_3x_4}{B_2} & -\frac{x_1x_2x_4}{B_3} & -\frac{x_1x_2x_3}{B_4} \\ \frac{x_2x_3+x_2x_4+x_3x_4}{B_1} & \frac{x_1x_3+x_1x_4+x_3x_4}{B_2} & \frac{x_1x_2+x_1x_4+x_2x_4}{B_3} & \frac{x_1x_2+x_1x_3+x_2x_3}{B_4} \\ -\frac{x_2+x_3+x_4}{B_1} & -\frac{x_1+x_3+x_4}{B_2} & -\frac{x_1+x_2+x_4}{B_3} & -\frac{x_1+x_2+x_3}{B_4} \\ \frac{1}{B_1} & \frac{1}{B_2} & \frac{1}{B_3} & \frac{1}{B_4} \end{pmatrix}. \tag{8}$$

The relations (6), (7) and (8) allow an immediate construction of the inverses of the Vandermonde matrices studied by Qi [14]. Turner [7] showed that V_n^{-1} is the product of two triangular matrices [15], which can be illustrated with the inverse matrices (6) and (7):

$$V_2^{-1} = \begin{pmatrix} 1 & -x_1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \frac{1}{B_1} & \frac{1}{B_2} \end{pmatrix}, V_3^{-1} = \begin{pmatrix} 1 & -\mathrm{x}_1 & \mathrm{x}_1\mathrm{x}_2 \\ 0 & 1 & -(\mathrm{x}_1 + \mathrm{x}_2) \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{\mathrm{x}_1-\mathrm{x}_2} & \frac{1}{\mathrm{x}_2-\mathrm{x}_1} & 0 \\ \frac{1}{\mathrm{B}_1} & \frac{1}{\mathrm{B}_2} & \frac{1}{\mathrm{B}_3} \end{pmatrix}. \tag{9}$$

3. Conclusion:

In this paper, we have presented a general and straightforward method for constructing the inverse of an arbitrary Vandermonde matrix with distinct parameters. The proposed approach is based on the characteristic polynomials of appropriately defined diagonal matrices, leading to explicit expressions for the entries of the inverse matrix. Unlike many existing inversion techniques, the present method provides a direct algebraic framework that is simple to apply and does not require recursive computations or matrix factorization algorithms.

The derived formulas are valid for Vandermonde matrices of arbitrary order and naturally reproduce the inverse matrices for lower-dimensional cases reported in the literature. Furthermore, the proposed formulation establishes an alternative approach to previously published algorithms while maintaining computational simplicity and mathematical transparency. Owing to its generality and explicit nature, the method may be useful in applications involving polynomial interpolation, numerical analysis, matrix computations, approximation theory, and other areas where Vandermonde matrices arise naturally. Future work may consider extending this approach to confluent Vandermonde matrices, generalized Vandermonde matrices, and related structured matrices that frequently occur in scientific computing and engineering applications.

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